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The forced Burgers equation, plant roots and Schrödinger's eigenfunctions

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Abstract. Unsaturated flow, in the presence of a web of plant roots, is modelled by Burgers' equation with a spatially varying sink function. The Hopf–Cole transformation results in a linear equation with non-constant coefficients. Separation of space and time variables leads to a stationary Schrödinger equation in which the analogue of potential energy is the integrated plant root water extraction rate. For balanced water supply on a finite domain, the basis of eigenfunctions is given explicitly for an inverse cube plant root density and for an exponentially decreasing plant root density.

Key words: forced Burgers, Schrödinger, plant roots

1. Introduction

Since machines with multi-megabyte information storage and multi-megaflop processing rates have become commonplace on desktops, and the prefix 'Giga' replaces 'Mega' when a parallel processor array is employed, number-crunching assaults on nonlinear partial differential equations will continue to be of practical importance. However, we must never overlook the benefits to be gained from exact solutions expressible in terms of familiar special functions. These provide insight on the relationships among physical variables, usually with an economy of effort. Adjustment of a parameter within an exact solution is often a trivial task compared to re-running a numerical algorithm after every change of parameter value. Exact solutions also serve as bench tests for the validation of numerical approximate solution algorithms. Finally, it must be admitted that some important practical P.D.E.s are out of reach of existing numerical schemes. For example, analytic methods may indicate the degree of singularities that may arise in temperature, concentration or concentration gradient predicted by reaction-diffusion equations or by nonlinear wave equations.

Much of our current store of exact solutions for interesting nonlinear P.D.E.s has been gleaned from ad-hoc methods that apply to restricted classes of P.D.E.s (*e.g.* [1, Chapter 1]). More catholic approaches include Lie-symmetry methods and their generalisations [2–12], singular manifold expansions based on the Painlevé criterion [13–17], equivalence methods for classes of P.D.E.s [18–19] and the method of differential constraints [20]. The whole subject has received an enormous boost in the latter 20th Century by the discovery of the inverse-scattering transform for soliton equations [17, 21–24].

From all of these analytic methods, humankind's conquests of practical nonlinear boundary value problems are rare enough in number to be contained in little more than a single tome [25]. This means that integrable P.D.E.s that are equivalent to linear equations either by change of variable (c-integrable [26, pp. 1–61]) or by the remarkable device of the inverse scattering transform (s-integrable [26, pp. 1–61]) are gems in the mullock heap. For after transforming an equation to a linear canonical form, we expect to be able to solve a variety of boundary value problems by using linear transforms.

Integrable nonlinear P.D.E.s have been systematically selected by the application of Painlevé tests [27–28], by existence of nonlinear superposition principles [29], by the criterion of infinite-dimensional Lie symmetry groups [30] and by the existence of a Lie-Bäcklund or extended (higher order) Lie-symmetry group [31–39]. The classification of second- and third-order integrable nonlinear scalar evolution equations is apparently complete [40, pp. 115–183]. For example Svinolupov [41] lists a full set of canonical forms, under the action of the group of contact transformations, for second-order evolution equations that possess a Lie-Bäcklund symmetry. These are

$$u_t = u_{xx} + g(x)u \quad \text{(linear class)}, \tag{1.1}$$

$$u_t = u_{xx} + 2uu_x + g(x) \quad \text{(Burgers class [42])}, \tag{1.2}$$

$$u_t = \partial_x[u^{-2}u_x + c_1 xu + c_2 u] \quad \text{(Fujita's equation [43])}, \tag{1.3}$$

$$\text{and } \; u_t = \partial_x[u^{-2}u_x] - 2 \quad \text{(Freeman-Satsuma equation [44])}, \tag{1.4}$$

where g is an arbitrary function and c_1, and c_2 are arbitrary constants. However the subject does not end there.

The main justification for the formal study of P.D.E.s is their ability to model and predict real behaviour of continuous matter, energy and force fields, as well as the more recent applications of financial derivatives (*e.g.* Wilmott *et al.* [45]) genetics, epidemiology and population dynamics (*e.g.* Britton [46]). In this area, much work remains to be done. If one applied an arbitrary contact transformation to the above canonical forms, then there would emerge an enormous variety of linearizable equations, including free parameters and free functions that can incorporate realistic models of experimentally realisable transport processes. The solution of practical nonlinear boundary-value problems all the way to the level of experimentally testable predictions, tends not to be a fashionable pursuit among mathematicians who naturally delight in building and consolidating the edifice of theory. However, this task is important because it is the ultimate justification for continued public support of mathematical analysis.

As detailed by Broadbridge *et al.* [47], each of the above equivalence classes has important applications in soil-water flow. However, after these general comments, attention will now be restricted to Burgers' equation with a source term, Equation (1.2). The original linearization procedure of Hopf [48] and Cole [49], previously known by Forsyth [50, p. 102, Ex.3], was applied to Burgers' equation without the source term $g(x)$. This conservative Burgers' equation has found many applications in gas dynamics, sedimentation theory and soil-water flow (Burgers [42], Hopf [48], Cole [49], Clothier *et al.* [51], Blake and Colombera [52], Sachdev [53], Broadbridge *et al.* [54]). This has prompted development of a general approach to solving associated canonical boundary-value problems [55] and has led to a catalogue of solutions [56]. The fact that Burgers' equation with a source term is linearizable has been independently discovered by many [53, 57–59]. However, explicit solutions to practical boundary-value problems involving the forced Burgers' equation are rare, and possibly non-existent in the literature.

For globally conservative problems, in which the balanced flux boundary conditions ensure that the x-integral of u over a finite domain $[0, \ell]$ is constant in time, a Sturm–Liouville problem results from the Hopf–Cole transformation [59]. However, it is not known for which forcing functions $g(x)$, the eigenfunctions can be constructed explicitly in terms of familiar special functions.

In Section 2, a model is developed for the transport of water through a field soil in the presence of a web of plant roots. In this case, a negative-valued x-dependent sink term represents a plant root extraction term averaged in the horizontal plane and diminishing with depth x into the soil where plant roots become sparse. In field soils which include biological macropores due to plant roots and wormholes, the soil water diffusivity typically increases only weakly with water content, and a linear diffusion term is adequate. A nonlinear convection term is necessary to ensure the development of a stable wetting front, as observed experimentally [60]. The Burgers equation with a sink term is an instructive model in this situation. The Hopf–Cole transformation leads to a stationary Schrödinger equation whose potential coincides with the potential for the plant-root forcing term. The Schrödinger operator is the archetypical self-adjoint operator. However, the standard quantum-mechanical stationary eigenstates originally constructed by Schrödinger cannot be used in this application. The harmonic oscillator potential cannot be used, as the plant-root sink term must approach zero as x tends to ∞. If a Coulomb potential is assumed, the stationary states of the H atom cannot be used, firstly because we have a second-derivative u_{xx} term rather than a Laplacian expressed in spherical coordinates, and secondly, because we have boundary conditions on a finite domain.

In Section 3, we consider a special inverse-cube forcing term. This leads to a simple basis of stationary eigenfunctions simply expressible in terms of trigonometric and polynomial functions. After separation of variables, it is seen that time dependence can be incorporated as exponential rather than oscillatory factors. Instead of a time-dependent wave equation, we have a dissipative diffusion equation which is analogous to a time-dependent Schrödinger equation with a pure imaginary Planck's constant.

The inverse-cube forcing term, which allows the Schrödinger equation to have additional Lie symmetries [61] and accidental degeneracy, is used in Section 2 merely as a device to obtain explicit solutions. A more realistic, more desirable and less restrictive model is the exponentially decreasing sink term with arbitrary plant-root length scale. In this case, considered in Section 3, the eigenfunctions are modified Bessel functions. In the case of balanced water supply, these Bessel functions have pure imaginary order. Solution of initial-boundary-value problems is much more difficult in this case, since one needs to find zeros of Bessel functions, viewed as functions of the complex valued order.

2. Unsaturated flow in the presence of plant roots

The nonlinear equation

$$\theta_t + \partial_z[K(\theta) - D(\theta)\theta_z] = -\Gamma g(z, \theta, t) \tag{2.1}$$

is widely accepted as a model for one-dimensional vertical unsaturated flow in soil with plant roots extracting water. Here, z is the depth beneath the soil surface, θ is volumetric water content, K is hydraulic conductivity, D is soil water diffusivity and Γg is the rate of water extraction by plant roots. For convenience, Γ is the extraction rate near the surface, where g

is unity. Γ^{-1} is a time scale for water extraction by roots. The term in square brackets in (2.1) is the volumetric water flux.

Lomen and Warrick [62] have solved linear water-extraction models with $K'(\theta)$ and D constant. Otherwise, most solutions to boundary value problems involving the nonlinear Equation (2.1) have been obtained numerically [63]. One exception is the integrable model of Broadbridge and Rogers [39], which has $(D, K, g) = (a(b-\theta)^{-2}, ma(b-\theta)^{-1}, \exp(-mz))$. They gave a solution procedure for the case of balanced constant-flux boundary conditions and arbitrary initial conditions. However, the solution was presented in full detail only in the steady state. For the time-dependent case of unbalanced water supply, the exact solution is still achievable, but it is extremely complicated (Li and Broadbridge, work in progress) and therefore of limited value.

In recompacted laboratory soils, $D(\theta)$ is typically a strongly increasing function. However, as explained by Clothier *et al.* [51], in field soils that contain many root channels and wormholes, D does not vary so much. Since $K(\theta)$, $K'(\theta)$ and $K''(\theta)$ are positive in real soils, we may represent $K(\theta)$ by a quadratic function

$$K = K_s\Theta^2 + K_n \quad \text{with } \Theta = \frac{\theta - \theta_n}{\theta_s - \theta_n},$$

where θ_s is water content at saturation, θ_n is some low background water content and $K_n = K(\theta_n)$. In many species, over a wide range of water contents, above the wilting point plant root extraction rate varies only weakly with soil water content [64]. Although explicit time dependence could be incorporated in the forced Burgers equation, we assume that for well established crops or plant communities, plant root density does not depend explicitly on time.

These considerations lead us to the weakly nonlinear Burgers equation with a force term,

$$\theta_t = D_*\theta_{zz} - 2K_s\frac{\theta - \theta_n}{(\theta_s - \theta_n)^2}\theta_z - \Gamma g(z/\ell_r), \tag{2.2}$$

where D_* is the representative soil-water diffusivity and ℓ_r is a plant root length scale. For convenience Γ is the water extraction rate near the surface and $g(0) = 1$.

D_* is chosen so that the model correctly predicts the measured sorptivity S. This requires [65]

$$D_* = \frac{\pi}{4}\frac{S^2}{(\theta_s - \theta_n)^2}. \tag{2.3}$$

Given uniform initial water content θ_n in a domain $[0, \infty)$ and constant concentration boundary condition $\theta(0, t) = \theta_s$ for $t > 0$, general nonlinear diffusion models predict the cumulative infiltration

$$i = \int_0^\infty (\theta - \theta_n)\mathrm{d}z + K_n t = St^{1/2} + O(t). \tag{2.4}$$

This behaviour is readily verified in the laboratory and in the field, and S can be accurately measured [66, pp. 187–208].

Now following Broadbridge and White [67], we define non-dimensional variables

$$\begin{aligned} \tau &= t/t_s \quad \text{with } t_s = \frac{\pi}{4}\frac{S^2}{(K_s - K_n)^2}, \\ \zeta &= z/\ell_s \quad \text{with } \ell_s = \frac{\pi}{4}\frac{S^2}{(K_s - K_n)(\theta_s - \theta_n)}. \end{aligned} \tag{2.5}$$

The time scale t_s is the time over which gravity begins to dominate capillary action in the transport of water from a saturated surface, when the $O(t)$ correction in (2.4) is no longer minor. This time scale may range from 30 minutes for a sand, up to a week for a clay [68]. The length scale ℓ_s is the capillary rise in a tube whose diameter is a typical soil pore diameter [69]. In terms of normalised water content Θ and dimensionless variables, the governing P.D.E. (2.2) is equivalent to

$$\Theta_\tau = \Theta_{\zeta\zeta} - 2\Theta\Theta_\zeta - \nu g(\zeta/\lambda_r), \tag{2.6}$$

where $\nu = \Gamma t_s/(\theta_s - \theta_n)$ and $\lambda_r = \ell_r/\ell_s$. This is the Burgers equation with an additional spatially varying sink term.

Applying the Hopf–Cole transformation

$$\Theta = -\frac{v_\zeta}{v}, \tag{2.7}$$

we obtain

$$\left[\frac{v_\zeta}{v} - \partial_\zeta\right]\left[v_\tau - v_{\zeta\zeta} - \nu\lambda_r G(\zeta/\lambda_r)v - h(t)v\right], \tag{2.8}$$

where function G is an integral of the function g and h is an arbitrary function. Hence it is sufficient that v satisfies a linear equation of the form

$$v_\tau - v_{\zeta\zeta} - \nu\lambda_r G(\zeta/\lambda_r)v = h(\tau)v \tag{2.9}$$

for some function h. Now consider separation of variables

$$v(\zeta,\tau) = p(\tau)q(\zeta),$$

leading to the separated system

$$p'(\tau) - h(\tau)p = -Ep, \tag{2.10}$$

$$-q''(\zeta) - \nu\lambda_r G(\zeta/\lambda_r)q = Eq. \tag{2.11}$$

Since (2.11) is the stationary Schrödinger equation with potential energy $-\nu\lambda_r G(\zeta/\lambda_r)$, there exists a basis of generalised eigenfunctions for $L^2(\mathbb{R})$. The force in this case is $-\nu g(\zeta/\lambda_r)$, which is attractive in our application, for which $g > 0$.

Over $L^2(\mathbb{R})$, the Schrödinger operator may have a continuous spectrum as well as a discrete spectrum, analogous to the ionised states and bound states of the H atom [70]. From (2.10) the general time-dependent solution to (2.9) is a linear combination of functions of the form

$$p_0 \exp\left(-E\tau + \int_0^\tau h(\tau_1)\,\mathrm{d}\tau_1\right) q_E(\zeta), \tag{2.12}$$

where $q_E(\zeta)$ are the independent eigenfunctions satisfying (2.11).

We now consider a finite layer of soil, with z restricted to $[0,\ell]$. Equivalently, ζ is restricted to $[0,\lambda]$ where $\lambda = \ell/\ell_s$. We assume that at the lower boundary there is an impermeable barrier, so that we have the no-flow boundary condition

$$\Theta^2 - \Theta_\zeta = 0 \quad \text{at } \zeta = \lambda. \tag{2.13}$$

This can be simply expressed in terms of the Hopf-Cole potential v as

$$\frac{v_{\zeta\zeta}}{v} = 0 \quad \text{at } \zeta = \lambda$$

implying

$$v_\tau(\lambda, \tau) = \left[h(\tau) + \frac{\nu \ell_r}{\ell_s} G(\lambda/\lambda_r)\right] v, \tag{2.14}$$

by (2.9). We are free to choose $h(\tau)$ to be the constant

$$h = -\nu\lambda_r G(\lambda/\lambda_r). \tag{2.15}$$

Then (2.14) reduces to

$$v(\lambda, \tau) = v_1, \tag{2.16}$$

for some constant v_1.

The total water extraction rate per unit cross section area is

$$\int_0^\ell \Gamma g(z/\ell_r)\,\mathrm{d}z.$$

In order for this to be balanced by the water flux at the boundary, we must have

$$K_s\Theta^2 - D_*\Theta_z = \int_0^\ell \Gamma g(z/\ell_r)\,\mathrm{d}z \quad \text{at } z = 0,$$

which implies

$$\Theta^2 - \Theta_\zeta = \nu\lambda_r[G(\lambda/\lambda_r) - G(0)] \quad \text{at } \zeta = 0. \tag{2.17}$$

Without detailed information on the structure of the extraction function $G(\zeta/\lambda_r)$ or on the initial water distribution $\Theta(\zeta, 0)$, the essential dimensionless parameters of the problem (2.6), (2.13), (2.17) are the dimensionless extraction rate ν, the dimensionless root length λ_r and the dimensionless layer thickness λ.

Applying the Hopf–Cole transformation (2.7) to (2.17), one obtains

$$v_{\zeta\zeta} = \nu\lambda_r[G(\lambda/\lambda_r) - G(0)]v \quad \text{at } \zeta = 0$$

implying $v_\tau = 0$, by (2.9) and (2.15).

Hence, v takes a constant value at $\zeta = 0$,

$$v(0, \tau) = v_0. \tag{2.18}$$

If we consider the case of a balanced water supply, then there may exist a steady state $v_s(\zeta)$ satisfying the governing equation and boundary conditions. Then $w(\zeta, \tau) = v(\zeta, \tau) - v_s(\zeta)$ will satisfy homogenous boundary conditions, along with the original governing equation. Separation of variables will then result in a Sturm–Liouville problem.

In the remaining sections, two choices of the water uptake function g lead to explicit presentation of the eigenfunctions in terms of familiar special functions and elementary functions. This will allow exact solution of the model of unsaturated flow with balanced flux boundary conditions, in the presence of a web of plant roots.

3. An elementary solution with inverse-cube sink term

Consider a plant-root extraction term $-\Gamma g(z/\ell_r)$ with

$$g(z/\ell_r) = \left(\frac{z}{\ell_r} + 1\right)^{-3}. \tag{3.1}$$

The stationary Schrödinger equation (2.11) then specialises to

$$-q''(\zeta) + \frac{\nu\lambda_r}{2}(\zeta/\lambda_r + 1)^{-2}q = Eq. \tag{3.2}$$

This is related to the equation for the radial Coulomb functions [71].

$$-q''(r) - \frac{aq}{r} + \frac{\ell(\ell+1)q}{r^2} = Eq,$$

wherein the inverse r term is the Coulomb potential and the inverse-square term in the effective potential is due to the quantised squared angular momentum $\ell(\ell + 1)$. An important difference here is that boundary conditions are imposed on a finite domain $[0, \lambda]$. Presumably, the eigenfunctions for the general form of (3.2) could be expressed in terms of hypergeometric functions. However, considerable simplicity can be gained after making some additional plausible restrictions. Firstly, the plant root length scale should be closely related to the intrinsic soil length scale [39], $\lambda_r = \ell_r/\ell_s = O(1)$, since capillary action in roots is analogous to capillary rise through soil pores. Indeed, the capillary length scale of a medium texture soil is typically 40 cm [68], a reasonable plant root depth. Secondly, it is assumed that the time scale $1/\Gamma$ for plant root uptake is closely related to the time $\ell_r\theta_s/K_s$ for water to fall through saturated soil to the depth of a typical plant root. Plant roots would not need to extend deeper in order to extract water efficiently. Hence it is reasonable to assume $\Gamma\ell_r/K_s = O(1)$. In the special case $\Gamma\ell_r/K_s = 4/\lambda_r^2$, (3.2) reduces to

$$(\zeta + \lambda_r)^2 q''(\zeta) - (2 - [\zeta + \lambda_r]^2 E)q = 0 \tag{3.3}$$

which has an elementary general solution

$$q = \frac{A}{\zeta + \lambda_r} + B(\zeta + \lambda_r)^2 \quad \text{for } E = 0, \tag{3.4}$$

$$\begin{aligned} q \;=\; & A\left(-\mu\sin(\mu[\zeta+\lambda_r]) - \frac{\cos(\mu[\zeta+\lambda_r])}{\zeta+\lambda_r}\right) \\ & + B\left(\mu\cos(\mu[\zeta+\lambda_r]) - \frac{\sin(\mu[\zeta+\lambda_r])}{\zeta+\lambda_r}\right) \quad \text{for } E = \mu^2 > 0, \end{aligned} \tag{3.5}$$

$$q = A\left(-\mu\sinh(\mu[\zeta+\lambda_r]) + \frac{1}{\zeta+\lambda_r}\cosh(\mu[\zeta+\lambda_r])\right)$$
$$+B\left(\mu\cosh(\mu[\zeta+\lambda_r]) - \frac{1}{\zeta+\lambda_r}\sinh(\mu[\zeta+\lambda_r])\right) \quad \text{for } E = -\mu^2 < 0. \qquad (3.6)$$

From (2.10)–(2.11), the steady state $v_s(\zeta)$ for $v(\zeta,\tau)$ is of the form (3.5) with $\mu = \sqrt{2}/(\lambda+\lambda_r)$. Since the gauge transformation $\bar{v} = v/A$ has no effect on the physical variable Θ, without loss of generality it may be assumed that in (3.5) $A = 1$ or $A = 0$. There remains a free parameter B which may in practice be determined by a physical quantity such as the total water volume [72].

Now the homogenised Hopf–Cole potential

$$w(\zeta,\tau) = v(\zeta,\tau) - v_s(\zeta) \qquad (3.7)$$

satisfies the linear Equation (2.9) subject to homogeneous boundary conditions

$$w(0,\tau) = w(\lambda,\tau) = 0. \qquad (3.8)$$

Hence, in accord with (2.12) we have an expansion of the form

$$v = v_s(\zeta) + \sum_{n=0}^{\infty} a_n \exp([h - E_n]\tau) q_n(\zeta) \qquad (3.9)$$

where $q_n(\zeta)$ are the orthonormal independent eigenfunctions of the form (3.4), (3.5) or (3.6) subject to boundary conditions (3.8). Since this is a regular Sturm–Liouville problem, there must exist a minimum eigenvalue E_0 and we may assume that E_n are in increasing order. In fact, there is a countable infinite set of eigenfunctions $q_n(\zeta)$ of the form (3.5), with $E_n = \mu_n^2 > 0$ obtained as positive roots of the equation

$$\tan\lambda\mu = \frac{\lambda\mu}{\mu^2\lambda_r(\lambda+\lambda_r)+1} = \phi(\mu) \qquad (3.10)$$

This is the condition that the two boundary conditions can be satisfied by nontrivial choice of A and B.

The curve $y = \phi(\mu)$ is tangent to the curve $y = \tan\lambda\mu$ at $\mu = 0$, has a single local maximum at

$$\mu = [\lambda_r(\lambda+\lambda_r)]^{-1/2},$$

a single positive valued inflection point at $\mu = [\lambda_r(\lambda+\lambda_r)/3]^{-1/2}$, and is asymptotic to $y = \lambda/\mu\lambda_r(\lambda+\lambda_r)$. Hence, $y = \phi(\mu)$ has one intersection with each continuous segment of the curve $y = \tan\lambda\mu$.

From the boundary conditions $q_n(0) = 0$, we deduce $(A, B) = (A_n, B_n)$, where

$$B_n = A_n \frac{\mu_n\lambda_r\tan(\mu_n\lambda_r)+1}{\mu_n\lambda_r - \tan(\mu_n\lambda_r)}.$$

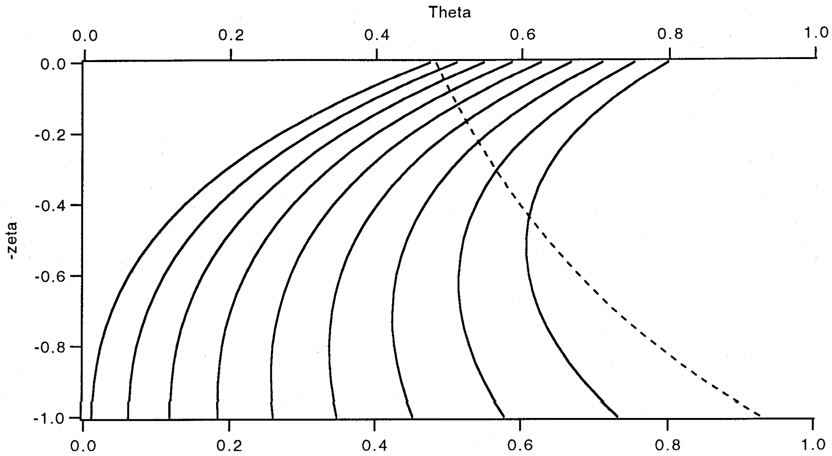

Figure 1. Steady state soil water profiles in the presence of plant roots. $\lambda = \lambda_r = 1$. Solid curves are steady states with total water content increasing from left curves to right curves. Dashed curve is the solution for conservative Burgers' equation in the absence of plant roots but with the same water content as rightmost solid curve.

Then (3.10) is the condition that $g_n(\lambda) = 0$. It is natural to choose A_n so that the eigenfunctions are normalized on $L^2(0, \lambda)$,

$$A_n^{-2} = \mu_n \int_{\lambda_r \mu_n}^{(\lambda+\lambda_r)\mu_n} \left\{ \sin\phi + \frac{\cos\phi}{\phi} + \frac{\mu_n \lambda_r \tan(\mu_n \lambda_r) + 1}{\mu_n \lambda_r - \tan(\mu_n \lambda_r)} \left(-\cos\phi + \frac{\sin\phi}{\phi} \right) \right\}^2 \mathrm{d}\phi.$$

Then, since the eigenfunctions of a self-adjoint operator are orthogonal, the solution for $v(\zeta, \tau)$ with initial data $v_0(\zeta)$ must be

$$v(\zeta, \tau) = v_s(\zeta) + \sum_{n=0}^{\infty} p_n \, \mathrm{e}^{(h-E_n)\tau} q_n(\zeta)$$

where

$$p_n = \int_0^{\lambda} q_n(\zeta) v_0(\zeta) \, \mathrm{d}\zeta.$$

For example, we consider uniform initial water content Θ_0. By (2.7), the initial condition for the Hopf-Cole potential may be taken to be

$$v_0(\zeta) = \exp(-\Theta_0 \zeta).$$

Figure 1 shows the steady-state moisture profiles with normalized water content Θ plotted against the dimensionless depth zeta. Each solution satisfies Burgers' equation with a sink (2.2) and (3.1), subject to zero flux at the lower boundary. The length parameters are chosen to be $\lambda = 1$ and $\lambda_r = 1$. There are many solutions because of the additional free parameter which may be regarded as total water content, increasing from left to right. For comparison, the steady-state solution of the conservative Burgers equation is included [73–74], with the same total water content as in the depicted nonconservative solution with the highest total water content. Whereas the conservative model predicts a water content increasing with depth,

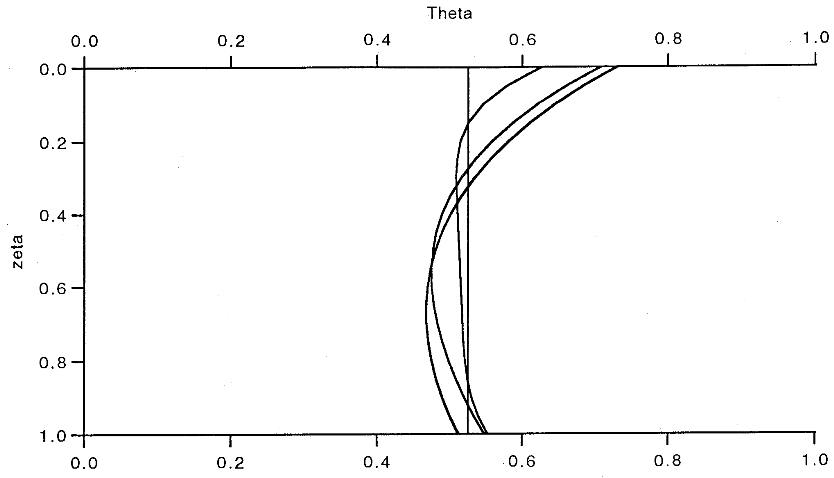

Figure 2. Evolving soil water profile in the presence of plant roots. Uniform initial water content $\Theta = 0{\cdot}53$, $\lambda = \lambda_r = 1$. Output times $\tau = 0{\cdot}0, 0{\cdot}01, 0{\cdot}1, 0{\cdot}5$ and $1{\cdot}0$.

the water consumption by plant roots demands a positive water flux at $\zeta = 0$, so that the water content must decrease below the surface. In the nonconservative solutions with high total water content, the local water content attains a minimum in the soil interior. Hence, the effect of plant roots on long-term water content profile can be quite marked.

Figure 2 displays a soil water content profile evolving from uniform initial conditions $\Theta = 0{\cdot}53$. The length parameters are again chosen to be $\lambda = \lambda_r = 1$. It is clear that the steady state has been effectively established at dimensionless time $\tau = 0{\cdot}5$, since the profiles at times 0·5 and 1·0 are practically indistinguishable.

4. Exponentially decreasing sink term

Now consider the case of an exponentially decreasing sink term,

$$g(z/\ell_r) = \exp(-z/\ell_r). \tag{4.1}$$

In this case, the Schrödinger equation (2.11) takes the form

$$-q''(\zeta) + \nu\lambda_r \exp(-\zeta/\lambda_r)q = Eq. \tag{4.2}$$

After changing independent variable to

$$\xi = 2\lambda_r^{3/2}\nu^{1/2}\exp(-\zeta/2\lambda_r),$$

Equation (4.2) transforms to

$$\xi^2 q_{\xi\xi} + \xi q_\xi - q(\xi^2 - \omega^2) = 0. \tag{4.3}$$

This is the modified Bessel equation of order $i\omega$, with general solution

$$q = AI_{i\omega}(\xi) + BK_{i\omega}(\xi). \tag{4.4}$$

Using the modified Bessel functions $K_{i\omega}$ of pure imaginary order, we may expand any function f in $L^2(0,\infty)$ as a continuous generalized Fourier expansion

$$f(\xi) = \frac{1}{\pi^2}\int_0^\infty K_{i\sqrt{\lambda}}(\xi)\sinh(\pi\sqrt{\lambda})\,\mathrm{d}\lambda \int_0^\infty K_{i\sqrt{\lambda}}(\zeta)\frac{f(\zeta)}{\zeta}\,\mathrm{d}\zeta. \tag{4.5}$$

This is the expansion formula of Section 4.15 of Titchmarsh [75] except that here the lower limit of integration is given correctly as 0 rather than $-\infty$.

When the elementary boundary conditions (3.8) are imposed on a finite domain, the continuous expansion (4.5) is replaced by a discrete orthonormal expansion in the same way as the standard Fourier transform would be replaced by a Fourier sine series. However, in this case the eigenvalues are $E_n = \omega_n^2/4\lambda_r^2$, where ω_n are the values that allow a nontrivial solution (A, B) of the system

$$AI_{i\omega_n}(\xi_0) + BK_{i\omega_n}(\xi_0) = 0, \qquad AI_{i\omega_n}(\xi_1) + BK_{i\omega_n}(\xi_1) = 0,$$

where $(\xi_0, \xi_1) = 2\lambda_r^{3/2}\nu^{1/2}(1, \exp(-\lambda/2\lambda r))$. That is, the ω_n values are zeros of modified Bessel functions, viewed as functions of the pure imaginary argument. As far as the author is aware, these have not been evaluated.

The steady state for $\Theta(\zeta)$ is given as

$$-\frac{1}{v_s}\frac{\mathrm{d}v_s}{\mathrm{d}\zeta} = \frac{\lambda_r^{1/2}\nu^{1/2}}{v_s}\exp(-\zeta/2\lambda_r)\frac{\mathrm{d}v_s}{\mathrm{d}\xi}$$

where $v_s = AI_{i\omega_0}(\xi) + BK_{i\omega_0}(\xi)$ with $\omega_0 = 2\lambda_r^{3/2}\nu^{1/2}\exp(-\lambda/2\lambda_r)$, by (2.12) and (2.15).

5. Conclusions

Although the Burgers equation with an appended spatially varying sink term is widely known to be exactly linearizable, the construction of exact solutions to practical boundary-value problems remains a challenging task. Here, we have concentrated on an application in soil water flow, in which the dependent variable Θ is the volumetric water content and the sink term represents extraction of water by plant roots. Applying separation of space and time variables, we find that the space-dependent factor satisfies a stationary linear Schrödinger equation whose potential energy is the integrated plant-root sink function, and the time-dependent factor is exponential. If we impose balanced flux boundary conditions on a finite domain, in which the net water supply balances the plant root requirements, then a steady-state solution exists. This steady state is relatively easy to calculate, as the Burgers equation then reduces to a Riccati equation, which is equivalent to a second-order linear ordinary differential equation. One of the two parameters of the general solution of this equation is redundant, as the water content Θ is not altered by a rescaling gauge transformation of the Hopf-Cole potential v. However, the second parameter allows us to freely vary a physical quantity such as the total water content of the finite soil column, which is invariant in the time-dependent solution. From the steady-state solutions it is already apparent that plant-root absorption may drastically modify the soil-water-content profile. The demands of plant roots must be in adaptive response to a long-term average water supply that can be maintained only with a predominantly negative water concentration gradient. This contrasts with the water build-up at the impermeable basement that will be more noticeable when plant roots are absent.

If we subtract the steady-state solution from the Hopf–Cole potential v, then there remains the task of solving a Sturm–Liouville eigenvalue problem. The form of the eigenfunctions is known explicitly for some special cases such as an inverse-cube sink function or a decreasing exponential sink function. For one special, but reasonable case of the inverse-cube sink functions, the eigenfunctions are elementary and the eigenvalues are solutions of a trigonometric-rational equation. In principle, since the Sturm–Liouville theory provides an orthonormal basis, the original boundary-value problem can be solved with any initial condition. For the exponential sink term, the eigenvalues remain to be evaluated as the zeros in the order domain of modified Bessel functions of pure imaginary order.

Although it has not been pursued further here, it is clear from (2.7) that constant-water-concentration boundary conditions on a finite domain will also lead to a Sturm–Liouville problem for the Hopf–Cole potential v, in this case satisfying linear boundary conditions of the Robin type. In this case, there will also exist a steady-state solution, and evaluation of the time-dependent solution will proceed in the same manner as for the previously discussed problem with balanced flux boundary conditions.

For the case of unbalanced flux boundary conditions, there will no longer be a steady-state solution. The boundary conditions for v will be time-dependent and no longer of the Sturm–Liouville class. Without a known solution satisfying the inhomogeneous boundary conditions, there is no ultimate gain in solving the related homogeneous problem. If we remove the time variable by applying the Laplace transform rather than by separating variables, then we will be faced with the problem of taking very difficult inverse Laplace transforms. For example, in the case of the exponential water-extraction term, we obtain a function in which the Laplace transform variable appears in the order of modified Bessel functions.

This example illustrates that idealized nonlinear boundary value problems are well worth studying, firstly, because they make contact with the real world and, secondly, because they generate intrinsically interesting and challenging mathematical questions. Intrinsically interesting mathematical questions generally have many unforeseen applications. Who would have thought that the spectral theory of Schrödinger operators, originally motivated by observations on atomic structure, would find future applications to waves in shallow water and to water uptake by plant roots?

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References

1. W. F. Ames, *Nonlinear Partial Differential Equations in Engineering*, Vol. 2. New York: Academic Press (1972) 305pp.
2. S. Lie, Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen. *Arch. Math.* 6 (1881) 328–368.
3. S. Lie, *Differentialgleichungen*. Leipzig (1891). Reprinted New York: Chelsea (1967) 561pp.
4. L. V. Ovsiannikov, *Gruppovye Svoystva Differentsialny Utraveni*. Novosibirsk: Soviet Academy of Sciences (1962) 406pp.

5. L. V. Ovsiannikov, *Group Analysis of Differential Equations*. English Edition, Boston: Academic Press (1982) 416pp.
6. G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations*. Berlin: Springer-Verlag (1974) 332pp.
7. J. M. Hill, *Solution of Differential Equations by Means of One-Parameter Groups*. Boston: Pitman (1982) 161pp.
8. L. Dresner, *Similarity Solutions of Nonlinear Partial Differential Equations*. London: Pitman (1983) 124pp.
9. P. J. Olver, *Applications of Lie Groups to Differential Equations*. New York: Springer-Verlag (1986) 513pp.
10. H. Stephani, *Differential Equations, their Solution using Symmetries*. Cambridge: Cambridge Univ. Press (1989) 260pp.
11. G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*. Berlin: Springer-Verlag (1989) 412pp.
12. N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol.1. Boca Raton: CRC Press (1994) 429pp.
13. J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations. *J. Math. Phys.* 24 (1983) 522–526.
14. J. Weiss, the Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative. *J. Math. Phys.* 24 (1983) 1405–1413.
15. A. C. Newell, M. Tabor and Y. B. Zeng, A unified approach to Painlevé expansions. *Physica* D29 (1987) 1–68.
16. W.-H. Steeb and N. Euler, *Nonlinear Evolution Equations and Painlevé test*. Singapore: World Scientific (1988) 333pp.
17. M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge: Cambridge University Press (1991) 516pp.
18. R. B. Gardner, *The Method of Equivalence and its Applications*. Philadelphia: Soc. for Industrial and Applied Math. (1989) 127pp.
19. J-F. Pommaret, *Partial Differential Equations and Group Theory: New Perspectives for Applications*. Boston: Dordrecht (1994) 473pp.
20. P. J. Olver, Direct reduction and differential constraints. *Proc. R. Soc. Lond.* A444 (1994) 509–523.
21. C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Method for solving the Korteweg-de Vries equation, *Phys. Rev. Lett.* 19 (1967) 1095–1097.
22. P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* 21 (1968) 467–490.
23. V. E. Zakharov and A. B. Shabat, A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. *Func. Anal. Appl.* 8 (1974) 226–235.
24. M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform - Fourier analysis for nonlinear problems. *Stud. Appl. Math.* 53 (1974) 249–315.
25. C. Rogers and W. F. Ames, *Nonlinear Boundary Value Problems in Science and Engineering*. New York: Academic Press (1989) 417pp.
26. F. Calogero, Why are certain nonlinear PDEs both widely applicable and integrable? In (V.E. Zakharov, (ed.)), *What is Integrability?* Berlin: Springer-Verlag, Berlin (1991) 321pp.
27. P. A. Clarkson, Painlevé analysis for the damped, driven, nonlinear Schrödinger equation. *Proc. Roy. Soc. Edin.* A109 (1988) 109–126.
28. N. Joshi, Painlevé property of general variable-coefficient versions of the Korteweg-de Vries and nonlinear Schrödinger equations. *Phys. Lett.* A125 (1987) 456–460.
29. J. M. Goard and P. Broadbridge, Nonlinear superposition principles obtained by Lie symmetry methods. *J. Math. Anal. Appl.* 214 (1997) 633–657.
30. G. W. Bluman and S. Kumei, Symmetry-based algorithms to relate partial differential equations: II. Linearization by nonlocal symmetries. *Euro. J. Appl. Math.* 1 (1990) 217–223.
31. P. J. Olver, Evolution equations possessing infinitely many symmetries. *J. Math. Phys.* 18 (1977) 1212–1215.
32. B. Fuchssteiner, Application of hereditary symmetries to nonlinear evolution equations. *Nonlinear Analysis Theory Method Appl.* 3 (1979) 849–862.
33. A. S. Fokas, A symmetry approach to exactly solvable evolution equations. *J. Math. Phys* 21 (1980) 1318–1325.
34. G. W. Bluman and S. Kumei, On the remarkable nonlinear diffusion equation $(\partial/\partial x)[a(u+b)^{-2}(\partial u/\partial x)] - (\partial u/\partial t) = 0$. *J. Math. Phys.* 21 (1980) 1019–1023.

35. A. S. Fokas and Y. C. Yortsos, On the exactly solvable equation occurring in two-phase flow in porous media. *SIAM J. Appl. Math.* 42 (1982) 318–332.
36. N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*. Dordrecht: Reidel (1985) 280pp.
37. P. Broadbridge, Integrable forms of the one-dimensional flow equation for unsaturated heterogeneous porous media. *J. Math. Phys.* 29 (1988) 622–627.
38. P. Broadbridge and S. E. Godfrey, Exact decaying soliton solutions of nonlinear Schrodinger equations: Lie-Bäcklund symmetries. *J. Math. Phys.* 32 (1991) 8–18.
39. P. Broadbridge and C. Rogers, On a nonlinear reaction-diffusion boundary-value problem: application of a Lie-Bäcklund symmetry. *J. Austral. Math. Soc.* B34 (1993) 318–332.
40. A. V. Mikhailov, A. B. Shabat and V. V. Sokolov, The symmetry approach to classification of integrable equations. In: V. E. Zakharov (eds) *What is Integrability?* Berlin: Springer-Verlag (1991) 321pp.
41. S. I. Svinolupov, Second-order evolution equations with symmetries. *Usp. Mat. Nauk.* 40 (1985) 263–264.
42. J. M. Burgers, A mathematical model illustrating the theory of turbulence. *Adv. Appl. Mech.* 1 (1948) 171–199.
43. H. Fujita, The exact pattern of a concentration dependent diffusion in a semi-infinite medium, part 2. *Textile Res. J.* 22 (1952) 823–827.
44. N. C. Freeman and J. Satsuma, Exact solutions describing an interaction of pulses with compact support in a nonlinear diffusive system. *Phys. Lett.* A138 (1989) 110–112.
45. P. Wilmott, S. Howison and J. Dewynne, *The Mathematics of Financial Derivatives*. Cambridge: Cambridge University Press (1995) 317pp.
46. N. F. Britton, *Reaction-Diffusion Equations and their Applications to Biology*. London: Academic Press (1986) 277pp.
47. P. Broadbridge, M. P. Edwards and J. E. Kearton, Closed-form solutions for unsaturated flow under variable flux boundary conditions. *Adv. Water Resour.* 19 (1996) 207–213.
48. E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.* 3 (1950) 201–230.
49. J. D. Cole, On a quasilinear parabolic equation occurring in aerodynamics. *Quart. Appl. Math.* 9 (1951) 225–236.
50. A. R. Forsyth, *Theory of Differential Equations*, Vol. VI. Cambridge: Cambridge University Press (1906) 582pp.
51. B. Clothier, J. H. Knight and I. White, Burgers' equation: application to field constant-flux infiltration. *Soil Sci.* 132 (1981) 255–267.
52. J. R. Blake and P. M. Colombera, Sedimentation: a comparison between theory and experiment. *Chem. Eng. Sci.* 32 (1977) 221–228.
53. P. L. Sachdev, *Nonlinear Diffusive Waves*. Cambridge: Cambridge University Press(1987) 246pp.
54. P. Broadbridge, R. Srivastava and T.-C. Jim Yeh, Burgers' equation and layered media: Exact solutions and applications to soil-water flow. *Math. Comp. Mod.* 16 (1992) 157–162.
55. F. Calogero and S. de Lillo, Burgers equation on the semiline with general boundary conditions at the origin. *J. Math. Phys.* 32 (1991) 99–105.
56. E. R. Benton and G. W. Platzman, A table of solutions of the one-dimensional Burgers equation. *Quart. Appl. Math.* 30 (1972) 195–212.
57. M. Moulana and G. A. Nariboli, On Bäcklund transformations of some 2nd-order nonlinear equations. *J. Math. Phys. Sci.* 11 (1975) 491–504.
58. J. J. C. Nimmo and D. G. Crighton, Bäcklund transformations for nonlinear parabolic equations: the general results. *Proc. R. Soc. Lond.* A384 (1982) 381–401.
59. R. A. Pasmanter, Stability and Bäcklund transform of the forced Burgers equation. *J. Math. Phys.* 29 (1986) 2744–2746.
60. J. R. Philip, Theory of infiltration. *Adv. Hydrosci.* 5 (1969) 215–296.
61. P. Rudra, Time dependent Schrödinger equation for a two-body system: Symmetry classes of central potentials. *Phys. Rev. Lett.* A41 (1990) 4042–4045.
62. D. O. Lomen and A. W. Warrick, Solution of the one-dimensional linear moisture flow equation with implicit water extraction functions. *Soil Sci. Soc. Am. J.* 40 (1976) 342–344.
63. F. J. Molz and I. Remson, Extraction term models of soil moisture use by transpiring plants. *Water Resour. Res.* 6 (1970) 1346–1356.
64. T. Talsma and E. A. Gardner, Soil water extraction by a moxed Eucalypt forest during a drought period. *Austr. J. Soil Res.* 24 (1986) 25–32.

65. J. R. Philip, The theory of infiltration: 4. Sorptivity and algebraic infiltration equations. *Soil Sci.* 84 (1957) 257–264.
66. I. White, M. J. Sully and K. M. Perroux, Measurement of surface-soil hydraulic properties: Disk permeameters, tension infiltrometers, and other techniques. In: G. C. Topp, W. D. Reynolds and R. E. Green (eds)) *Advances in Measurement of Soil Physical Properties: Bringing Theory into Practice*. Madison: Soil Sci. Soc. Am.(1992) 288pp.
67. P. Broadbridge and I. White, Constant rate rainfall infiltration: A versatile nonlinear model, 1. Analytic solution. *Water Resour. Res.* 24 (1988) 145–154.
68. I. White and P. Broadbridge, Constant rate rainfall infiltration: A versatile nonlinear model, 2. Applications of solutions. *Water Resour. Res.* 24 (1988) 155–162.
69. I. White and M. J. Sully, Macroscopic and microscopic capillary length and time scales from field infiltration. *Water Resour. Res.* 23 (1987) 1514–1522.
70. J. von Neumann, *Mathematical Foundations of Quantum Mechanics*. Princeton: Princeton University Press (1955) 445pp.
71. L. I. Schiff, *Quantum Mechanics,* 2nd ed. New York: McGraw-Hill (1955) 417pp.
72. W. W. Read and P. Broadbridge, Series solutions for steady unsaturated flow in irregular porous domains. *Transp. Porous Media* 22 (1996) 195–214.
73. P. Broadbridge and C. Rogers, Exact solutions for vertical drainage and redistribution in soils. *J. Eng. Math.* **24** (1990) 25–43.
74. A. W. Warrick and G. W. Parkin, Analytical solution for one-dimensional drainage: Burgers' and simplied forms. *Water Resour. Res.* 31 (1995) 2891–2894.
75. E. C. Titchmarsh, *Eigenfunction Expansions, Part 1,* 2nd ed. Oxford: Clarendon Press (1962) 203pp.